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A note on the viscous diffusion of rolled vortex sheets

By J. P. GUIRAUD
Laboratoire de Mécanique Théorique, associé au CNRS, Université de Paris VI, Tour 66, Place Jussieu, 75230 Paris Cedex, 05, and ONERA, 92320 Chatillon, France

AND R. KH. ZEYTOUNIAN
U.E.R. de Mathématiques Pures et Appliquées, Université de Lille I, B.P. 36, 59650 Villeneuve d'Ascq and ONERA, 92320 Chatillon, France

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It is well known that vortex sheets are diffused by viscosity. For a plane sheet this diffusion is described by a linear diffusion equation. Here we consider a rolled vortex sheet, and show that the viscous diffusion of the vorticity concentrated at the turns of the sheet, when they are very closely spaced, can also be described by a linear equation, in appropriately transformed variables.

1. Introduction

A double-scale technique was successfully applied by Guiraud & Zeytounian (1977, which we shall refer to hereafter as GZ) in order to describe analytically the asymptotic structure of the core of a rolled vortex sheet in an inviscid, incompressible, irrotational flow. In the present note, we use a slight extension of this technique to derive a linear equation which describes, quantitatively, the laminar viscous diffusion of the closely spaced turns of the highly rolled part of the sheet.

We start with the Navier–Stokes equations for incompressible flow in dimensionless form. Let $\mathbf{u}$ be the dimensionless velocity, p the dimensionless pressure and t and $\mathbf{x}$ the dimensionless time and space variables. Let $R = UD/\nu$ be the Reynolds number, where D is a length scale of the order of the diameter of the core, U is a velocity scale of the order of the azimuthal component of the velocity, and ν is the kinematic viscosity. The azimuthal component is the component of velocity which is normal to the meridian plane in local cylindrical co-ordinates with respect to the axis of the core. t, $\mathbf{x}$ and $\mathbf{u}$ are of course non-dimensionalized with respect to DU^{-1}, D and U.

The technique used in GZ was to regard $\mathbf{u}$ and p as functions of t and $\mathbf{x}$, considered as slow variables, and also of another variable χ, a fast one, which was itself a function of t and $\mathbf{x}$. The role of the fast variable χ was to describe the rapid variation of $\mathbf{u}$ and p transverse to the sheet. It was found, as expected, that this rapid variation had a saw-tooth structure. The corresponding discontinuities were found to be associated with the vorticity concentrated at the turns of the sheet. The saw-tooth signature was normalized by prescribing that χ changes by 2π from one discontinuity to the next, i.e. between two consecutive turns of the sheet. χ was of course a multi-valued function of $\mathbf{x}$.

Let us denote by e a length scale which is of the order of the distance between two consecutive turns of the sheet, in the main part of the core. We define a closeness parameter

$$C = e/D \tag{1}$$

and a viscous parameter

$$\eta = C^2 R. \tag{2}$$

We assume that $C \ll 1$ and $R \gg 1$ in such a way that $\eta = O(1)$ or smaller. Within the core we search for an asymptotic representation of the flow

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{u}^*(t, \mathbf{x}; \chi; C; \eta) = \mathbf{u}_0^* + C\mathbf{u}_1^* + \ldots, \\ p &= p^*(t, \mathbf{x}; \chi; C; \eta) = p_0^* + Cp_1^* + \ldots, \\ \boldsymbol{\Omega} &= \boldsymbol{\Omega}^*(t, \mathbf{x}; \chi; C; \eta) = \boldsymbol{\Omega}_0^* + C\boldsymbol{\Omega}_1^* + \ldots, \end{aligned}\right\} \tag{3}$$

where $\boldsymbol{\Omega}$ is the vorticity. We observe that from the definition of χ we may write

$$\partial\chi/\partial t = C^{-1}\omega, \quad \nabla\chi = C^{-1}\mathbf{k}, \tag{4}$$

where ω and $|\mathbf{k}|$ are $O(1)$. Equations (4) and the hypothesis that $C \ll 1$ make it clear that χ is a fast variable.

The main result of the calculation presented below is twofold. On the one hand $\mathbf{u}_0^*$ and p_0^*, which are found to be independent of χ, are also found to be solutions of the inviscid equations. On the other hand it is shown that the first approximation $\boldsymbol{\Omega}_0^*$ to the vorticity may be expressed as

$$\boldsymbol{\Omega}_0^* = 2\pi\phi(T, \chi)\, \nabla \wedge \mathbf{u}_0^*, \tag{5}$$

where ϕ is a solution of the heat equation

$$\partial\phi/\partial T = \partial^2\phi/\partial\chi^2. \tag{6}$$

Here T is a pseudo-time and is defined by

$$T = \eta^{-1}\int_{t_0}^{t} |\mathbf{k}_1|\, dt_1, \tag{7}$$

where the integration is effected along the trajectories of the velocity field $\mathbf{u}_0^*$ and $\mathbf{k}_1$ stands for the value of $\mathbf{k}$ at t_1 along such a trajectory.

Of course, on physical grounds such a result is not unexpected. As was mentioned to the authors by a referee, this result was obtained on intuitive grounds by Moore & Saffman (1973). Nevertheless, it does not seem likely that (6), complemented by (7), could be derived in a formal way as simply as is done in the present work without applying the double-scale technique which is used here.

A comment may be helpful for the reader: the vorticity concentrated on the rolled sheet in the inviscid model is weak, its intensity being $O(C)$, as shown in GZ, and this is the main reason why (6) is linear.

2. Formal derivation

We substitute (3) into the Navier–Stokes equations, observe that, for example,

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{u}^*}{\partial t} + C^{-1}\omega\frac{\partial \mathbf{u}^*}{\partial \chi} \tag{8}$$

and equate like powers of C to zero. In fact, besides the Navier–Stokes equations, it proves convenient to use the equation defining the vorticity $\mathbf{\Omega}$, namely

$$\mathbf{\Omega}^* = \nabla \wedge \mathbf{u}^* + C^{-1}\mathbf{k} \wedge (\partial \mathbf{u}^*/\partial \chi). \tag{9}$$

Through a straightforward argument, using the fact, readily proved, that $\partial \mathbf{\Omega}_0^*/\partial \chi$ cannot be identically zero, we get to leading order

$$\partial \mathbf{u}_0^*/\partial \chi = \partial p_0^*/\partial \chi = \omega + \mathbf{k}\,.\,\mathbf{u}_0^* = 0. \tag{10}$$

Considering then the next order, we find that there are terms secular with respect to χ in $\mathbf{u}_1^*$ etc. These secular terms must be rejected, and in order to achieve this, we require that $\mathbf{u}_1^*$, etc. be periodic functions of χ with period 2π. The choice of 2π for the period is just one of normalization. Dealing with periodic functions of the variable χ, it proves useful to set

$$\overline{f^*} = (2\pi)^{-1}\int_0^{2\pi} f^*(\chi)\,d\chi, \quad \tilde{f}^*(\chi) = f^*(\chi) - \overline{f^*}, \tag{11}$$

so that $\overline{f^*}$ is the average of f^* over one period, while the average of $\tilde{f}^*$ is zero. Writing down the equations to one order higher to the one leading to (10) and averaging them with respect to χ, we obtain the conditions for elimination of the secular terms in $\mathbf{u}_1^*$ etc.:

$$\nabla\,.\,\mathbf{u}_0^* = 0, \tag{12a}$$

$$\partial \mathbf{u}_0^*/\partial t + (\mathbf{u}_0^*\,.\,\nabla)\,\mathbf{u}_0^* + \nabla p_0^* = 0, \tag{12b}$$

$$\overline{\partial \mathbf{\Omega}_0^*/\partial t} + (\mathbf{u}_0^*\,.\,\nabla)\,\overline{\mathbf{\Omega}_0^*} - \overline{(\mathbf{\Omega}_0^*}\,.\,\nabla)\,\mathbf{u}_0^* = 0, \tag{12c}$$

$$\overline{\mathbf{\Omega}_0^*} = \nabla \wedge \mathbf{u}_0^*. \tag{12d}$$

We then find that, for example,

$$\mathbf{u}_1^* = -|\mathbf{k}|^{-2}\,\mathbf{k} \wedge \mathbf{V}_0^*, \quad \partial \mathbf{V}_0^*/\partial \chi = \tilde{\mathbf{\Omega}}_0^*. \tag{13}$$

Here we should mention that (12c) is derived from the vorticity equation and emphasize that the viscous term has disappeared as a consequence of the averaging process. Returning to the vorticity equation expanded to the order which leads to (12c) and subtracting the latter from the former, we get

$$\begin{aligned} -\mathbf{k}\,.\,(\nabla \wedge \mathbf{u}_0^*)\,|\mathbf{k}|^{-2}\,\mathbf{k} \wedge \tilde{\mathbf{\Omega}}_0^* = {} & \partial \tilde{\mathbf{\Omega}}_0^*/\partial t \\ & + (\mathbf{u}_0^*\,.\,\nabla)\,\tilde{\mathbf{\Omega}}_0^* - (\tilde{\mathbf{\Omega}}_0^*\,.\,\nabla)\,\mathbf{u}_0^* - \eta^{-1}|\mathbf{k}|^2\,\partial^2 \tilde{\mathbf{\Omega}}_0^*/\partial \chi^2. \end{aligned} \tag{14}$$

Now we observe that (14) has been derived under the assumption $\eta = O(1)$, in order to describe the situation when the rolled sheet has been diffused by viscosity to the extent that the thickness of each sheet is becoming of the order of the spacing between the turns. But one may convince oneself that the expansion with respect to C is a time-like expansion corresponding to the process of following a particle as it progresses towards the core of the rolled sheet, while R remains of fixed order of magnitude. As a consequence, letting $\eta \to \infty$ corresponds to enforcing an initial condition for the diffusion process. In the limit we must recover the inviscid theory considered in GZ; then for almost all χ we have $\tilde{\Omega}_0^* = -\overline{\Omega_0^*}$ from irrotationality, except in vanishingly thin viscous mixing layers. Under such conditions, (12c) shows that the right-hand side of (14) must be zero. As a consequence we conclude that in the limit $\eta \to \infty$ we must have

$$\mathbf{k}\,.\,(\nabla \wedge \mathbf{u}_0^*) = 0. \tag{15}$$

Now, if we consider that the equations defining $\mathbf{u}_0^*$ and p_0^* do not depend on η, we conclude that (15) must hold whatever the value of η, and this leads to the following equation:

$$\partial\tilde{\mathbf{\Omega}}_0^*/\partial t + (\mathbf{u}_0^* \cdot \nabla)\tilde{\mathbf{\Omega}}_0^* - (\tilde{\mathbf{\Omega}}_0^* \cdot \nabla)\mathbf{u}_0^* = \eta^{-1}|\mathbf{k}|^2\partial^2\tilde{\mathbf{\Omega}}_0^*/\partial\chi^2. \tag{16}$$

The last step in the derivation of our main result is to go from (16) to (5) and (6) complemented by (7). The trick which achieves this goal in the simplest way is to use a Lagrangian description of the flow associated with $\mathbf{u}_0^*$. We define the tensor F by

$$d\mathbf{x} = \mathsf{F} \cdot d\mathbf{a} + \mathbf{u}_0^*\, dt, \quad \mathbf{x} = \mathbf{X}(t, \mathbf{a}), \tag{17}$$

where $\mathbf{a}$ is a Lagrangian variable, and denote the inverse of F by G. Then standard arguments lead to Cauchy's equation for the vorticity in a perfect fluid, which can be used to derive from (16) the equation†

$$\partial(\mathsf{G} \cdot \tilde{\mathbf{\Omega}}_0^{*L})/\partial T - \partial^2(\mathsf{G} \cdot \tilde{\mathbf{\Omega}}_0^{*L})/\partial\chi^2 = 0, \tag{18}$$

where the pseudo-time T is defined according to

$$T = \eta^{-1}\int_{t_0}^{t} |\mathbf{k}(\tau, \mathbf{X}(\tau, \mathbf{a}))|^2\, d\tau, \tag{19}$$

which is nothing other than (7).

We now make a final reduction of (18). This is achieved through use of Cauchy's vorticity relation

$$\partial\{\mathsf{G} \cdot (\nabla \wedge \mathbf{u}_0^*)\}/\partial t = 0, \tag{20}$$

which shows that the formula

$$\mathbf{\Omega}_0^* = 2\pi\phi(T, \chi)\, \nabla \wedge \mathbf{u}_0^* \tag{21}$$

agrees with (16) provided that ϕ satisfies the heat equation (6).

That (21) is the proper solution of (16), taking into account (12d), is seen by observing that $\eta \to \infty$ is equivalent to $T \to 0$ and that for $\eta \to \infty$ we get (21), where ϕ is an infinite sum of Dirac delta functions concentrated on $\chi = (2K+1)\pi$, K running through positive and negative integers.

3. Conclusion

We have shown that the diffusion of a rolled vortex sheet may be described by the heat equation. We need only to define properly the variable χ transverse to the sheets and to use a proper pseudo-time T. Then the process of diffusion is a universal one, and throughout this process the averaged flow $\mathbf{u}_0^*$ remains inviscid and rotational. In order to obtain some effect of viscosity on the averaged flow we must go to very high values of T, which means that we must go very deep into the rolled sheet towards the axis of the core. We refer to Stewartson & Hall (1963) for the description of a continuous core flow diffused by viscosity.

† The superscript L means that we use as independent variables t and $\mathbf{a}$ instead of t and $\mathbf{x}$, as well as χ.

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